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On the monopole and quantization of charge

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Abstract : The motion of an electron in the magnetic field of a monopole is investigated, both for the Dirac and the Klein-Gordon equations. It is shown that the single-valuedness of the angular factor of the wave function in both the cases lead to the quantization of charge

Keywords : Magnetic monople, quantization of charge, relativistic wave equations.

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1. Introduction

The object of this short paper is to investigate the relativistic motion of a charge particle in the field of a magnetic monopole in the rest frame of the monopole. Dirac [1(i)] predicted the existence of magnetic monopole following the analysis of the nonintegrability of the phase of the (quantum) wave function. This leads in a natural manner, to the quantisation of charge, *e.g.* $\frac{e\mu}{\hbar c} = \frac{1}{2}$, ($\mu$-pole-strength). It is shown in the present paper that the single-valuedness of the wave function of the charge particle in the magnetic field of the monopole leads directly to the quantization of charge. We obtain the same rule for quantization as that of Dirac (1/2 integer) on considering the Dirac equation for an electron in the field of a monopole as well as the Klein-Gordon equations for an electron in the field of a monopole. In a subsequent paper, Dirac [1(ii)] considered the motion of a magnetic monopole in the field of an electron. The equation of motion of the monopole is taken as that of Dirac equation, supplemented with the subsidiary equation for the electromagnetic potential A^μ to be imposed to admit the existence of free magnetic monopoles.

2. The potential

The magnetic field $\boldsymbol{B}$ and the vector potential $\boldsymbol{A}$ due to an (isolated) pole of strength μ is

$$\boldsymbol{B} = \mu\, r/r^3 \text{ and } \boldsymbol{A} = \mu(1-\cos\theta)\,\nabla\varphi \tag{1}$$

in spherical polar co-ordinates (r,θ,φ) with the pole as the centre. The constant of integration is taken as μ, so that the nodal line is $\theta = 0$, to $\theta = \pi$, (Dirac [1(ii)], Harish Chandra [2]).

3. The Dirac equation

The Dirac equation for the electron is

$$\left\{ i\hbar\frac{\partial}{c\partial t} + \alpha\left(i\hbar\nabla - \frac{e}{c}\boldsymbol{A}\right) - \beta mc \right\}\psi(t,\boldsymbol{r}) = 0 \tag{2}$$

Writing

$$\psi(t,r) = \exp-\left(\frac{i}{\hbar}Et + \frac{\beta\chi}{2}\right)\psi(\boldsymbol{r}), \tag{3}$$

where

$$\tanh\chi = \frac{mc^2}{E} \tag{4a}$$

*Deceased

and

$$\hbar M = \left(\frac{E^2}{c^2} - m^2c^2\right)^{1/2} . \tag{4b}$$

The equation for $\psi(r)$ in which β no longer appears, is [3(ii)]

$$\{\alpha(\nabla + igA) - iM\}\psi(\boldsymbol{r}) = 0, \tag{5}$$

$\left(g = \frac{e\mu}{\hbar c}\right)$. This equation in terms of spherical polar coordinate, with

$$\psi(\boldsymbol{r}) = \left\{\exp\left(\frac{\alpha_x\alpha_y}{2} + in\right)\varphi\right\}\psi(r,\theta) \tag{6}$$

$$\left[-iM + \alpha_z\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{\alpha_x}{r}\left\{\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right.\right.$$

$$\left.\left. + i\alpha_x\alpha_y(n + g - g\cos\theta)/\sin\theta\right\}\right]\psi(r,\theta). \tag{7}$$

It needs to point out that for single valued $\psi(\boldsymbol{r})$,

$$n + \frac{1}{2} = \text{an integer.} \tag{8}$$

(Note that the eigenvalues of $\alpha_x\alpha_y$ are $\pm i$). Next, $i\alpha_x\alpha_y\alpha_z$ commutes with the operator on $\psi(r,\theta)$ in eq. (7). Hence, we can take

$$i\alpha_x\alpha_y\alpha_z = \varepsilon = \pm 1. \tag{9}$$

Let

$$i\alpha_x\alpha_y U_\pm = \pm U_\pm \equiv \varepsilon\alpha_z U_\pm$$

$$\text{and} \quad \alpha_x U_\pm \equiv U_\mp \equiv i\alpha_y U_\pm , \tag{10}$$

($U_\pm$-each two dimensional). The eq. (7) suggests that $\psi(r,\theta)$ may be written in the form :

$$r\psi(r,\theta) = R_+(r)S_+(\theta)U_+ + R_-(r)S_-(\theta)U_- \tag{11}$$

($R_\pm$,$S_\pm$ are scalar functions). The equations for $R_\pm$ (r) and $S_\pm$ (θ) are

$$rR_-(r)^{-1}\left(\varepsilon\frac{d}{dr} - iM\right)R_+(r)$$

$$+ S_+(\theta)^{-1}D_-(\theta)S_-(\theta) = 0, \tag{12a}$$

$$rR_+(r)^{-1}\left(\varepsilon\frac{d}{dr} + iM\right)R_-(r)$$

$$- S_-(\theta)^{-1}D_+(\theta)S_+(\theta) = 0, \tag{12b}$$

where

$$D_\pm(\theta) = \frac{d}{d\theta} + \left(\frac{1}{2} \mp g\right)\cot\theta \pm (n+g)\cos\theta. \tag{13}$$

3.1. The angular functions :

From eqs. (12), it follows that

$$D_-(\theta)S_-(\theta) = \Lambda_+ S_+(\theta) \tag{14a}$$

and

$$D_+(\theta)S_+(\theta) = -\Lambda_- S_-(\theta). \tag{14b}$$

where $\Lambda_\pm$ are constants. So that equations for $S_\pm$ ($u = \cos\theta$) are :

$$\left[(1-u^2)\frac{d^2}{du^2} - 2u\frac{d}{du} - \frac{1}{1-u^2}\right.$$

$$\times\left\{(n+g)^2 + \left(g \mp \frac{1}{2}\right)^2 - 2(n+g).\right.$$

$$\left.\left.\left(g \mp \frac{1}{2}\right)u\right\} + \Lambda_+\Lambda_-\right]S_\pm(u) = 0. \tag{15}$$

Let

$$S_\pm(u) = (1+u)^{p_\pm}(1-u)^{q_\pm}P_\pm(u) , \tag{16}$$

where

$$2p_\pm^2 + 2q_\pm^2 = (n+g)^2 + (g \mp \tfrac{1}{2})^2 \tag{17a}$$

and

$$2q_\pm^2 - 2p_\pm^2 = -2(n+g)(g \mp \tfrac{1}{2}). \tag{17b}$$

The equation for $P_\pm(u)$ is then given by

$$\left[(1-u^2)\frac{d^2}{du^2} + 2\{p_\pm - q_\pm - (p_\pm + q_\pm + 1)u\}\frac{d}{du}\right.$$

$$\left. + \Lambda'_\pm\right]P_\pm(u) = 0 \tag{18}$$

where

$$\Lambda'_\pm = \Lambda_+\Lambda_-(p_\pm + q_\pm)^2 - (p_\pm + q_\pm). \tag{19}$$

These equations have bounded solutions only when

$$\Lambda'_\pm = k(k + 2p_\pm + 2q_\pm + 1), \tag{20}$$

where k is any +ve integer and the solutions are well-known Jacobi Polynomials $P_k^{2p,2q}(u)$, (Smirnov [4]).

3.2. The quantization :

The expressions for $S_\pm(\theta)$ are given by eqs. (16–20) as

$$S_\pm(\theta) = 2^{2p_\pm+2q_\pm}\left(\cos\frac{\theta}{2}\right)^{2p_\pm}.$$

$$\sin\frac{\theta}{2}\Big| \quad . P_k^{2p_\pm,2q_\pm}(\cos\theta). \tag{21}$$

Thus, $S_\pm(\theta)$ are single-valued and bounded only when $2p_\pm$ and $2q_\pm$ are +ve integers. Finally, one obtains from eqs. (17)

$$2q_\pm = n \pm \frac{1}{2} \tag{22}$$

and

$$2p_\pm = \left|\left|n \pm \frac{1}{2}\right| \pm 2|g|\right| \tag{23}$$

+ or − in between in eq. (23) are according as $n \pm 1/2$ and g are both of the same sign or one of them is opposite to the other. Since $2p \pm$ and $n \pm 1/2$ are integers, one concludes that $2g$ should also be an integer.

Thus,

$$N = 2g = \frac{2e\mu}{\hbar c} \text{ and } \frac{e\mu}{\hbar c} = \frac{1}{2}(N = 1), \tag{24}$$

(N is an integer). Thus, the smallest unit of $\frac{e\mu}{\hbar c} - \frac{1}{2}$ which leads to quantization of charge e, more precisely of '$e\mu$'.

3.3. The radial functions :

From eqs. (12), it follows

$$r\left(\varepsilon\frac{d}{dr} - iM\right)R_+(r) + \Lambda_+ R_-(r) = 0, \tag{25a}$$

$$r\left(\varepsilon\frac{d}{dr} + iM\right)R_-(r) - \Lambda_- R_+(r) = 0. \tag{25b}$$

Hence,

$$\left\{\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \pm i\varepsilon M + \Lambda_+\Lambda_- + M^2\right\}R_\pm = 0. \tag{26}$$

This equation shows that both R_+ and R_- tends to $\exp \pm iMr$ as $r \to \infty$. M is real for positive energy of the electron ($M^2 > 0$, eq. (4)). So that the radial part of the wave functions as $r \to \infty$, becomes $\psi(r) \propto (1/r) \exp \pm iMr$. Thus, $|\psi(r)|^2$ taken over a large spherical surface ($r \to \infty$) is a nonvanishing constant. Hence, the electron is not bound to the pole as is expected from the classical theory (Poincaré [5]). The wave functions spread over to infinity. Setting aside this factor $\exp \pm iMr$, the remaining factor of $R_\pm(r)$ can easily be expressed in an ascending power of r, which is conditionally convergent with respect to the parameters M and $\Lambda_+\Lambda_-$. Further, since $\Lambda_+\Lambda_- \neq 0$ both the solutions satisfy the boundary condition $r\psi(r) \to 0$ as $r \to 0$.

4. The Klein-Gordon equation

With the potential A given by the eq (1), the Klein-Gordon equation for the problem [6] in spherical-polar coordinates is given by

$$\Big[M^2 + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\times\left\{\frac{\partial^2}{\partial\varphi^2} + i2g(1-\cos\theta)\frac{\partial}{\partial\varphi} - g^2(1-\cos\theta)^2\right\}\Big]\psi(\boldsymbol{r}) = 0. \tag{27a}$$

Thus,

$$\psi(\boldsymbol{r}) = F(r)G(\theta)e^{in\varphi} \tag{27b}$$

for unique wave function $\psi(\boldsymbol{r})$, n = integer. (28)

The equation for $G(\theta)$ expressed in terms of $u = \cos\theta$, is

$$\Big[\frac{d}{du}(1-u^2)\frac{d}{du} - \frac{1}{(1-u^2)}\left\{n^2 + 2(ng+g^2) - 2g(n+g)u\right\} + \Lambda\Big]G(u) = 0 \quad (\Lambda \text{ is constant}) \text{ and} \tag{29}$$

that for $F(r)$

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{1}{r^2}(g^2 - \Lambda) + M^2\right\}F(r) = 0. \tag{30}$$

4.1. The Angular function G(u) :

As before, taking

$$G(u) = (1+u)^p(1-u)^q P_0(u), \tag{31}$$

where

$$2p^2 + 2q^2 = (n + g)^2 + g^2 \tag{32a}$$

and

$$2q^2 - 2p^2 = -2(n + g)g \tag{32b}$$

the equation for $P_0(u)$

$$\left[\left(1-u^2\right)\frac{d^2}{du^2}+2\{p-q-(p+q+1)u\}\frac{d}{du}+\Lambda'\right]P_0(u)$$

$$= 0, \tag{33}$$

where

$$\Lambda' = \Lambda-(p+q)^2-(p+q). \tag{34}$$

Eq. (32) for $P_0(u)$ has bounded solution in the interval $(-1 \leq u \leq 1)$, only when

$$\Lambda' = k(k+2p+2q+1), \tag{35}$$

where k is any +ve integer and the solutions are well known Jacobi Polynomials $P_k^{2p,2q}(u)$ [4].

4.2. The quantization :

$$G(\theta) = 2^{2(p+q)}\left(\cos^{2p}\frac{\theta}{2}\right)\left(\sin^{2q}\frac{\theta}{2}\right)P_k^{2p,2q}(\cos\theta) \tag{36}$$

Thus, $G(\theta)$ is single-valued and bounded only when $2p$ and $2q$ are +ve integers. Finally, one obtains from eq. (31),

$$2q = |n| \tag{37}$$

and

$$2p = \big\| n | \pm 2 | g \big\| \tag{38}$$

+ or − in eq. (38) according as n and g are both of the same sign or one of them is of opposite sign. Since $2p$ and n are integers, we finally obtain that $2g$ should also be an integer :

$$2g = N \text{ and } \frac{e\mu}{\hbar c} = \frac{1}{2}(N = 1) \tag{39}$$

(N any interger). Thus, the smallest unit $\frac{e\mu}{\hbar c} = \frac{1}{2}$, leading to the quantization of charge.

4.3. The radial function :

The eq. (30) for $F(r)$ shows that as $r \to \infty$, $rF(r) \to \exp \pm iMr$, for positive energy $M^2 > 0$ [eq. (4)] (M is real). As before, in this case also, the charged particle is not bound to the pole; as expected from classical theory, it goes over to infinity.

5. Discussion

It needs to be emphasized that our result is susceptible to the choice of constant (gauge?) for the potential eq. (1), following Dirac [1(ii)]. In our investigation this appears to be the supplementary condition on the potential. Further, it may be pointed out that the result, *i.e.*, the quantization of the charge is independent of the spin of the electron, as both the Dirac equation and the Klein-Gordon equation lead to the same result.

References

[1] P A M Dirac *(i) Proc Roy Soc.* **A133** 60 (1931); *(ii) Phys Rev* **74** 817 (1948)

[2] Harish-Chandra *Phys Rev* **74** 883 (1948)

[3] N D Sen Gupta *(i) Nuovo. Cim Sr* **X42** 121 (1966), *(ii) Phys. Lett.* **A152** 453 (1991)

[4] W I Smirnov *Lehrgang der Höheren Mathematik III* (1) (1959)

[5] H Poincaré *Compt. Rend.* **123** 930 (1896)

[6] I Tamm *Zeit. F. Physik* **71** 141 (1931) (This paper is devoted mostly a generalization of spherical Harmonics)